Effect of finite boundaries on the Stokes resistance of an arbitrary particle

By HOWARD BRENNER
Department of Chemical Engineering, New York University

(Received 10 March 1961 and in revised form 5 July 1961)

A general theory is put forward for the effect of wall proximity on the Stokes resistance of an arbitrary particle. The theory is developed completely for the case where the motion of the particle is parallel to a principal axis of resistance. In this case, the wall-effect correction can be calculated entirely from a knowledge of the force experienced by the particle in an *unbounded* fluid, providing (i) that the wall correction is already known for a spherical particle and (ii) that the particle is small in comparison to its distance from the boundary. Experimental data are cited which confirm the theory. The theory is extended to the wall effect on a particle *rotating* near a boundary.

1. Introduction

The surprisingly large effects of wall proximity on the Stokesian resistance of a settling particle are well known. To cite an illustration of Birkhoff (1950) '...a sphere falling slowly in a cylindrical tube of viscous liquid, having 100 times the cross-section of the sphere, encounters 20 % more resistance than if there were no walls'. Because of the practical importance of low Reynolds number drag phenomena in applications, numerous theoretical studies of the problem have been carried out. However, with few exceptions (Wakiya 1957, 1959; Chang 1961), these investigations have been limited to spherical particles. Particles encountered in practice are rarely of this shape and it is therefore desirable to have available the analogous wall-effect corrections for non-spherical particles.

It often suffices to know the magnitude of this correction only for relatively small ratios of the characteristic particle-to-wall dimensions. As will be shown, this objective may be achieved without further effort whenever the analogous correction is already known for a spherical particle.

Attention is confined to situations where sufficient symmetry prevails to cause the force on the settling body to act parallel to its direction of motion. The necessary and sufficient conditions for this are: (i) the *particle* must be moving parallel to one of its three principal axes of resistance (see § 2 for the definition of this term), and (ii) the *boundary* must possess three mutually perpendicular symmetry planes orthogonal to the principal axes of the particle. These requirements are not as restrictive as might otherwise appear. Virtually all important applications fall within their purview.

Where applicable, the central result of our analysis is remarkably simple. Let D denote the drag on the particle when moving in the bounded medium with velocity U, and let D_∞ denote the drag on the particle when moving through the unbounded fluid at the same velocity. The correction to the Stokes law resistance is then of the form

$$\frac{D}{D_\infty} = \frac{1}{1-k(D_\infty/6\pi\mu Ul)+O(c/l)^3}, \tag{1.1}$$

in which c and l are, respectively, the characteristic particle and wall dimensions, and μ is the viscosity of the fluid. The dimensionless constant k is independent of the shape of the particle, depending solely on the nature of the bounding wall. The value of k can be obtained at once by comparing equation (1.1) to the known solution of the problem for a spherical particle of radius c, in which case $D_\infty = 6\pi\mu cU$.

The existence of a general relation of the type (1.1) is suggested by the recent work of Chang (1961) who showed that the drag on any body of revolution, falling parallel to its symmetry axis at the centre of a circular cylinder of radius l, is given by the expression

$$\frac{D}{D_\infty} = 1+2{\cdot}1044\frac{D_\infty}{6\pi\mu Ul}+O\left(\frac{c}{l}\right)^2. \tag{1.2}$$

When applied to a circular cylinder, equation (1.1) leads to results of greater accuracy than can be obtained from the above. Moreover, it shows that the assumption of axisymmetric motion required in Chang's derivation is unduly restrictive.

It is possible to obtain results analogous to equation (1.1) which are free from all symmetry restrictions. These results are, however, of a decidedly less elementary nature than those given here and are not further discussed.

2. Particle settling in a stationary fluid

Consider a particle, P, settling with instantaneous velocity $\mathbf{U}$ near some surface, S. It is assumed that the fluid motion is governed by the creeping motion and continuity equations

$$\nabla^2\mathbf{v} = \mu^{-1}\nabla p, \quad \nabla . \mathbf{v} = 0. \tag{2.1, 2.2}$$

We confine ourselves to the case in which the particle and wall are solid surfaces, to which fluid adheres. (In a subsequent paragraph we shall remove this restriction on S.) The boundary conditions to be satisfied are then

$$\mathbf{v} = \mathbf{U} \quad \text{on } P, \quad \mathbf{v} = \mathbf{0} \quad \text{on } S. \tag{2.3, 2.4}$$

Since the net flow of fluid is zero we also require that†

$$\mathbf{v} \to \mathbf{0} \quad \text{as} \quad r \to \infty, \tag{2.5}$$

where r is measured from the 'centre' of the particle; that is, from the point of intersection of its three principal axes.

† In the event that S is a closed surface containing P in its interior, this condition need not be satisfied.

As the equations of motion and boundary conditions are linear, we assume that the fluid velocity and pressure fields can be decomposed into a sum of fields

$$\mathbf{v} = \mathbf{v}^{(1)}+\mathbf{v}^{(2)}+\mathbf{v}^{(3)}+\mathbf{v}^{(4)}+\dots, \tag{2.6}$$

$$p = p^{(1)}+p^{(2)}+p^{(3)}+p^{(4)}+\dots, \tag{2.7}$$

each term of which, $(\mathbf{v}^{(n)}, p^{(n)})$, separately satisfies the equations of motion. The individual fields are to be determined successively by application of the following boundary conditions

$$\mathbf{v}^{(1)} = \mathbf{U} \quad \text{on } P, \tag{2.8}$$

$$\mathbf{v}^{(2)} = -\mathbf{v}^{(1)} \quad \text{on } S, \tag{2.9}$$

$$\mathbf{v}^{(3)} = -\mathbf{v}^{(2)} \quad \text{on } P, \tag{2.10}$$

$$\mathbf{v}^{(4)} = -\mathbf{v}^{(3)} \quad \text{on } S, \text{ etc.} \tag{2.11}$$

and, in addition, for $n = 1, 2, 3, \dots,$

$$\mathbf{v}^{(n)} \to \mathbf{0} \quad \text{as} \quad r \to \infty. \tag{2.12}$$

This techique of solution is known as the 'method of reflexions' (Brenner & Happel 1958), each separate field being termed a reflexion from either P or S. As the odd numbered fields involve the satisfaction of a boundary condition on P, they introduce the characteristic particle dimension, c, through terms of the form c/r raised to some positive power. Likewise, since $r = O(l)$ on S, the even numbered fields, determined by the boundary conditions on S, introduce its characteristic dimension, l. Thus, each successive pair of reflexions, $P \to S \to P$, increases the overall accuracy of the solution by contributing terms in c/l whose dominant powers are of higher order than those arising from the preceding reflexions. Equations (2.6) and (2.7) therefore amount to a series expansion in ascending powers of c/l which converges to the solution of the original boundary-value problem posed.

The drag, $\mathbf{D}$, exerted *on* the particle *by* the fluid, may be obtained by summing the drag contributions of each of the individual fields. As shown in the Appendix, no contribution to the drag is made by the even numbered fields, reflected from S, and thus

$$\mathbf{D} = \mathbf{D}^{(1)}+\mathbf{D}^{(3)}+\mathbf{D}^{(5)}+\dots. \tag{2.13}$$

The initial field, $\mathbf{v}^{(1)}$, obviously corresponds to the settling of a particle in an unbounded fluid. Associated with this motion is the drag

$$\mathbf{D}^{(1)} = \mathbf{D}_\infty, \tag{2.14}$$

where, as before, $\mathbf{D}_\infty$ refers to the force exerted *by* the fluid *on* the particle. A detailed knowledge of $\mathbf{v}^{(1)}$ is required only in so far as it determines the next reflexion, $\mathbf{v}^{(2)}$, through the boundary condition on S, equation (2.9). As this boundary is situated at a relatively great distance from the particle, we may take the initial field to be that generated by a *point force*, $\mathbf{D}_\infty$, situated at the centre of the particle (Lamb 1932)

$$\mathbf{v}^{(1)} = -\frac{\mathbf{D}_\infty}{6\pi\mu r} - \frac{r^2}{24\pi\mu}\nabla(\mathbf{D}_\infty . \nabla)\frac{1}{r}, \tag{2.15}$$

and

$$p^{(1)} = \frac{1}{4\pi}(\mathbf{D}_\infty . \nabla)\frac{1}{r}. \tag{2.16}$$

It is helpful to bear in mind that $\mathbf{D}_\infty$ is proportional to c. For a particle of finite dimensions these equations are correct to terms involving higher powers of c/r than those already implicit in them.

To the degree of approximation to which equation (2.15) is valid, the initial field is independent of the shape of the particle, being determined entirely by its resistance. This property obviously carries over to the next field, $\mathbf{v}^{(2)}$, defined by equation (2.9). The field $\mathbf{v}^{(2)}$ depends, of course, explicitly on the geometry of S. This second field makes no contribution to the drag and is important to our calculations only in so far as it determines the third field, $\mathbf{v}^{(3)}$, in accordance with equation (2.10).

Assuming $\mathbf{v}^{(2)}$ to be known, we are now in a position to calculate the drag contribution $\mathbf{D}^{(3)}$. The calculation does not require an *a priori* knowledge of $\mathbf{v}^{(3)}$. We seek here an analogue of Faxén's sphere theorem (Oseen 1927, p. 113; Péres 1929), by means of which the force experienced by a 'small' particle, suspended in an arbitrary field of flow, can be computed. This can be accomplished with the aid of a reciprocal theorem due to Lorentz (Villat 1943) which, in our present application, takes the form

$$\iint_P d\mathbf{S}.\mathbf{\Pi}^{(1)}.\mathbf{v}^{(3)} = \iint_P d\mathbf{S}.\mathbf{\Pi}^{(3)}.\mathbf{v}^{(1)}, \tag{2.17}$$

where $\mathbf{\Pi}^{(1)}$ and $\mathbf{\Pi}^{(3)}$ denote the pressure tensors associated with the corresponding velocity fields and $d\mathbf{S}$ is a directed element of surface area normal to the particle, P, over whose surface the integration extends.†

At the surface of the particle we have that $\mathbf{v}^{(1)} = \mathbf{U}$, whereupon the right-side of equation (2.17) becomes $\mathbf{U}.\mathbf{D}^{(3)}$, in which $\mathbf{D}^{(3)} = \iint_P d\mathbf{S}.\mathbf{\Pi}^{(3)}$ is the force on the particle due to $\mathbf{v}^{(3)}$.

In the left-hand integral of equation (2.17), we have from equation (2.10) that $\mathbf{v}^{(3)} = -\mathbf{v}^{(2)}$ at the surface of the particle. As the field $\mathbf{v}^{(2)}$ is regular within the region of space presently occupied by the particle, it can be expanded in a Taylor series about the centre of the particle. If only the leading term in the expansion is retained, this yields $\mathbf{v}^{(3)} = -\mathbf{v}_0^{(2)}$ on P, in which the subscript 0 implies that the field is to be evaluated at the centre of the particle. Retention of only the leading term in the series expansion at the surface of the particle depends, ultimately, for its justification on the fact that c/l is small. By these means, the remaining integral in equation (2.17) becomes $-\mathbf{v}_0^{(2)}.\,\mathbf{D}_\infty$, where

$$\mathbf{D}_\infty = \mathbf{D}^{(1)} = \iint_P d\mathbf{S}.\mathbf{\Pi}^{(1)}.$$

Upon collecting results, we are led to the expression

$$-\mathbf{v}_0^{(2)}.\mathbf{D}_\infty = \mathbf{U}.\mathbf{D}^{(3)}, \tag{2.18}$$

† Equation (2.17) requires that the surface integrals vanish when extended over the surface of an indefinitely large sphere containing the particle in its interior. This is easily seen to be the case here as we have for either field that $\mathbf{v} \sim O(1/r)$ and $p \sim O(1/r^2)$, as $r \to \infty$. Inasmuch as $\mathbf{\Pi} = -\mathbf{I}p + \mu(\nabla\mathbf{v} + \mathbf{v}\nabla) \sim O(1/r^2)$ and $d\mathbf{S} \sim O(r^2)$, it follows at once that the integrals vanish appropriately.

correct to terms involving higher powers of c/l than those already implicit in the relation. Now, in general, for any particle in the Stokes régime, we have the following linear relation (Landau & Lifshitz 1959) connecting the force experienced by the particle and its velocity,

$$\mathbf{D}_\infty = -\mathbf{\Phi}.\mathbf{U}. \tag{2.19a}$$

Here, $\mathbf{\Phi}$ is a symmetric resistance tensor. It is independent of the orientation of the particle with respect to its direction of motion through the fluid. It follows from the above relations that

$$\mathbf{D}^{(3)} = \mathbf{\Phi}.\mathbf{v}_0^{(2)}. \tag{2.19}$$

This relation constitutes the generalization of Faxén's law.

The properties of symmetric tensors are such that every *arbitrary* particle in creeping flow is endowed with a unique set of three mutually perpendicular axes such that, if its motion through the unbounded fluid be parallel to one of them, it will experience a force only in this direction. We shall refer to these as principal axes of resistance. When a body (e.g. an ellipsoid) possesses three mutually perpendicular symmetry planes, its principal axes of resistance lie normal to them. Attention is confined in the sequel to motion parallel to a principal axis. Thus, when the particle moves through the unbounded fluid with velocity $\mathbf{U} = \mathbf{k}U$, parallel to any of these axes, only one component of force results, $\mathbf{D}_\infty = -\mathbf{k}D_\infty$, in a direction opposite to that of the particle motion. From equation (2.15) we then find that

$$\mathbf{v}^{(1)} = \frac{D_\infty}{8\pi\mu}\left[\mathbf{i}\frac{xz}{r^3}+\mathbf{j}\frac{yz}{r^3}+\mathbf{k}\left(\frac{1}{r}+\frac{z^2}{r^3}\right)\right],$$

where (x, y, z) are measured from the centre of the particle, along its principal axes.

To establish the direction of $\mathbf{v}_0^{(2)}$, we note that, when the three symmetry planes of the boundary lie normal to the principal axes of the particle, the only non-zero component of $\mathbf{v}_0^{(2)}$ will be parallel to the direction of motion of the particle. This follows from equation (2.10), and the above expression for $\mathbf{v}^{(1)}$, by observing that this is the only velocity component of the field $\mathbf{v}^{(2)}$ which will not be an *odd function* of at least one of the co-ordinates—x, y or z. Furthermore, any components which are odd functions vanish at the centre of the particle ($x = 0, y = 0, z = 0$). Thus we write $\mathbf{v}_0^{(2)} = -\mathbf{k}v_0^{(2)}$, where the scalar $v_0^{(2)}$ may be either positive or negative.

It follows then from equation (2.19) that $\mathbf{D}^{(3)}$ is parallel to the direction of motion of the particle. Thus, if we write $\mathbf{D}^{(n)} = -\mathbf{k}D^{(n)}$, equation (2.19) adopts the form

$$D^{(3)} = D_\infty v_0^{(2)}/U, \tag{2.20}$$

in which D_∞ and U are essentially positive. The algebraic sign of $D^{(3)}$ is then positive or negative according as $v_0^{(2)}$ is positive or negative. For later reference, we also write $\mathbf{D} = -\mathbf{k}D$ as it is now obvious that the *total* force on the particle is parallel to its direction of motion and oppositely directed. The scalar D is essentially positive.

As $\mathbf{D}^{(3)}$ is now known, the velocity field $\mathbf{v}^{(3)}$ can be computed at large distances from the particle via an equation of the form (2.15), in which $\mathbf{v}^{(3)}$ replaces $\mathbf{v}^{(1)}$, and $\mathbf{D}^{(3)}$ replaces $\mathbf{D}_\infty$. This then furnishes the boundary conditions to determine $\mathbf{v}^{(4)}$ in accordance with equation (2.11). By analogy to our previous calculations, it is at once apparent that $D^{(5)} = D_\infty[v_0^{(2)}/U]^2$, $D^{(7)} = D_\infty[v_0^{(2)}/U]^3$, etc. Upon summing the individual drags in accordance with equation (2.13), the following expression is obtained for the total drag on the particle:

$$D = D_\infty + D_\infty[v_0^{(2)}/U] + D_\infty[v_0^{(2)}/U]^2 + D_\infty[v_0^{(2)}/U]^3 + \ldots.$$

This geometric series may be summed, whereupon

$$\frac{D}{D_\infty} = \frac{1}{1-(v_0^{(2)}/U)}. \tag{2.21}$$

From equation (2.15), the initial field, $\mathbf{v}^{(1)}$, is proportional to D_∞. Since $\mathbf{v}^{(2)}$ is linearly connected to $\mathbf{v}^{(1)}$ through equation (2.9), it follows that the same must be true of $\mathbf{v}^{(2)}$. In addition, it is clear that $\mathbf{v}^{(2)}$ must be independent of the viscosity of the fluid. Since D_∞ is directly proportional to the viscosity, this requires that $\mathbf{v}^{(2)}$ be proportional to D_∞/μ. Finally, $\mathbf{v}^{(2)}$ must obviously go to zero as S recedes infinitely far from P, i.e. as $l \to \infty$. By simple dimensional arguments, it follows that we must have

$$v_0^{(2)} = k(D_\infty/6\pi\mu l), \tag{2.22}$$

the dimensionless constant k depending solely on the nature of the boundary S. The boundary correction is then of the form

$$\frac{D}{D_\infty} = \frac{1}{1-k(D_\infty/6\pi\mu U l)}. \tag{2.23}$$

It remains yet to estimate the degree of approximation inherent in the above relation. Since D_∞ is proportional to $\mu U c$, the result is certainly correct to first powers of c/l, so that the error cannot exceed terms of $O[(c/l)^2]$ in the denominator of (2.23). However, other arguments, too lengthy to give here, suggest that the error does not really exceed terms of $O[(c/l)^3]$. In this event, we are led to equation (1.1) as the correct form of the drag correction.

It is worthwhile noting that equation (1.1) is correct whether or not S is a solid surface to which fluid adheres. Careful re-examination of the previous development shows that the final result holds for any *linear* boundary conditions on S, such as, for example, that for a 'free' surface, i.e. one on which the normal velocity and tangential stresses vanish. It goes without saying that the value of k depends upon the boundary conditions imposed on S.

Equation (1.1) can be extended to the case where the fluid itself is in a state of *net* flow as, for example, when fluid flows through a tube or between parallel walls within which a particle is confined. This is done by the addition of a field $(\mathbf{v}^{(0)}, p^{(0)})$, corresponding to flow through the conduit in the absence of the particle. The sole modification required in equation (1.1) is the replacement of the denominator of the left-hand side by D'_∞, the infinite-medium drag based on the *approach velocity* to the particle, i.e.

$$D'_\infty = D_\infty[1-(v_0^{(0)}/U)\{1+O(c/l)^2\}]. \tag{2.24}$$

3. k-coefficients for typical boundaries

The numerical values of k required to complete equation (1.1) can be obtained for any boundary where the wall effect is already known for a spherical particle. In this case equation (1.1) becomes

$$\frac{D}{6\pi\mu c U} = \frac{1}{1-k(c/l)+O(c/l)^3}. \tag{3.1}$$

Numerous solutions involving spherical particles are available in the literature with which equation (3.1) can be compared to determine the appropriate k value. These have been employed to obtain the values cited below.

Case I. Particle moving along the axis of a circular cylinder; l = radius of cylinder (Oseen 1927, p. 198; Haberman & Sayre 1958):

$$k = 2{\cdot}1044. \tag{3.2}$$

Case II. Particle at the centre of a hollow sphere; l = radius of sphere (Cunningham 1910; Haberman & Sayre 1958):

$$k = \tfrac{9}{4}. \tag{3.3}$$

Case III. Particle falling perpendicular to a single, infinite, plane surface; l = distance from centre of particle to plane:

(*a*) 'solid' plane (Oseen 1927, p. 142), $k = \frac{9}{8}$; (3.4)

(*b*) 'free' surface,† $k = \frac{3}{4}$. (3.5)

Case IV. Particle moving parallel to a single, infinite, plane surface; l = distance from centre of particle to plane:

(*a*) 'solid' plane (Oseen 1927), $k = \frac{9}{16}$; (3.6)

(*b*) 'free' surface,‡ $k = -\frac{3}{8}$. (3.7)

Case V. Particle falling midway between two infinite, plane, parallel walls and moving parallel to them; l = distance from centre of particle to either wall (Oseen 1927, p. 204):

$$k = 1{\cdot}004. \tag{3.8}$$

Case VI. Same as case I, except that centre of particle is situated at a fractional distance $\beta = b/l$ from cylinder axis (Brenner & Happel 1958; Famularo 1961):

$$k = f(\beta), \tag{3.9}$$

where $f(\beta)$ is defined by Brenner & Happel in their equation (A2). For small values of β they give

$$f(\beta) = 2{\cdot}1044 - 0{\cdot}6977\beta^2 + O(\beta^4). \tag{3.10}$$

† This value is derived from Faxén's solution (Oseen 1927, p. 161) for two equal spheres *approaching* one another, along their line of centres, with equal velocities, by observing that the plane midway between them is a free surface.

‡ This value is derived from Smoluchowski's solution (Oseen 1927, p. 204) for two equal spheres falling with equal velocity perpendicular to their line of centres, by observing that the plane midway between them is a free surface.

It appears from the recent numerical work of Famularo (1961) that their tentative plot of $f(\beta)$, for larger values of β, is somewhat in error. More accurate values, obtained by Famularo through direct evaluation of the integral of Brenner & Happel, are as follows:

β	0·0	0·2	0·4	0·6	0·8	0·9
$f(\beta)$	2·104	2·079	2·044	2·165	$3{\cdot}213 \pm 0{\cdot}3$	$5{\cdot}60 \pm 0{\cdot}5$

As the particle approaches the cylinder wall, i.e. $\beta \to 1$, the solution reduces to that for a particle settling next to a plane wall. Hence,

$$\lim_{\beta \to 1} (1-\beta) f(\beta) = \tfrac{9}{16}, \tag{3.11}$$

provided that the fractional distance of the particle from the cylinder wall, $c/(l-b)$, is small. Thus, equations (6.8) and (9.5) of Brenner & Happel are in error. Equation (3.11) is consistent with Famularo's (1961) numerical results.

4. Experimental confirmation

The predictions of the present theory can be compared to the experimental data of Squires & Squires (1937) on the low Reynolds number fall of circular disks along the axis of a circular cylinder. These authors measured the wall-effect correction for disks moving both broadside-on and edge-on in the range $0{\cdot}16 > c/l > 0{\cdot}08$. Their data for D/D_∞ correspond to our case I, with $D_\infty = 16\mu c U$ and $(32/3)\mu c U$, respectively (Lamb 1932). A comparison of their experimental data (their figure 2) with the present theory is made below, in figure 1. The solid lines correspond to the theoretical values. Agreement appears to be well within the accuracy of their data.

The extensive low Reynolds number settling experiments ($Re < 0{\cdot}05$) of Pettyjohn & Christiansen (1948) carried out with cubes, octahedra and tetrahedra falling symmetrically along the axis of an 8 in. diameter circular cylinder provide a further test of the theory, as do the analogous experiments of Heiss & Coull (1952) on the fall of finite cylinders and rectangular parallelepipeds of various aspect ratios along each of their sets of principal axes in a 7·09 cm circular cylinder. Theoretical values are *not* known for the resistances of these particles in infinite media. These authors give experimental plots of the dimensionless parameter $K = 18\mu U/(d_s^2 g \Delta\rho)$ *vs* d_s (in cm) in the range $0 < d_s/2l < 0{\cdot}10$. Here, d_s is the diameter of a spherical particle of equal volume, g is the acceleration of gravity and $\Delta\rho$ is the density difference between particle and fluid. According to equations (1.1) and (3.2), with $D/D_\infty = U_\infty/U$ (U_∞ = settling velocity of the particle in the unbounded fluid), $K = K_\infty - 2{\cdot}1044 d_s/2l$. Thus, a plot of K *vs* d_s ought to yield a straight line with a negative slope of $1{\cdot}0522/l$. The data of Pettyjohn & Christiansen (1948) (their figure 6) show a series of straight parallel lines for each of the four shapes investigated, their average slope being $0{\cdot}112 \pm 0{\cdot}010\,\text{cm}^{-1}$. This compares favourably with the theoretical slope of $0{\cdot}104\,\text{cm}^{-1}$. The data of Heiss & Coull (1952) (their figure 1) also appear as a series of straight parallel lines with an average slope of $0{\cdot}292 \pm 0{\cdot}005\,\text{cm}^{-1}$, in excellent agreement with the theoretical value of $0{\cdot}297\,\text{cm}^{-1}$.

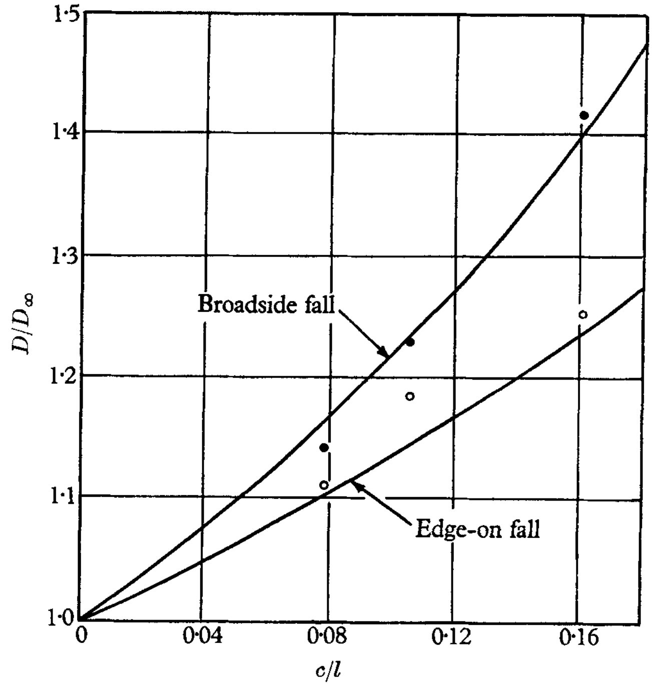

FIGURE 1. Wall effect for circular disks falling along the axis of a circular cylinder. ●, Broadside fall; ○, edge-on fall.

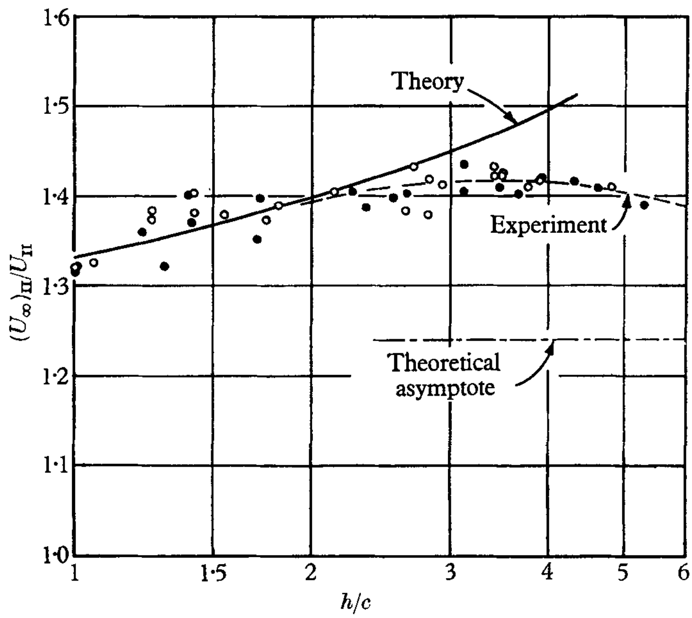

FIGURE 2. Wall effect for two spheres falling parallel to their line-of-centres along the axis of a circular cylinder. ○, $\frac{1}{2}$ in. Lucite spheres; ●, $\frac{1}{2}$ in. Marbelette spheres.

Further experimental verification of the general theory is provided by the experimental data of Pfeffer (1958) on the fall of two closely spaced, equal sized spheres of the same density following one another along the axis of a circular cylinder at small Reynolds numbers. In interpreting Pfeffer's (1958) results, we adopt the point of view that two neighbouring particles may be looked upon as a single, dumbbell-like body which experiences a drag equal to the total drag on the two particles, and whose 'centre' lies along the line-of-centres joining the particles. This interpretation seems indisputable when the particles are actually in contact as well as when the centre-to-centre spacing, $2h$, is small compared to the cylinder radius l. On the other hand, it is obvious that the interpretation will lead to errors when h/l is large. It follows from equation (1.1) that

$$\frac{D_{\mathrm{II}}}{(D_\infty)_{\mathrm{II}}} = \frac{1}{1-2{\cdot}104\,(D_\infty)_{\mathrm{II}}/6\pi\mu Ul},$$

where D_{II} and $(D_\infty)_{\mathrm{II}}$ are the total drags on both spheres when they move with velocity U within the cylinder and in the infinite fluid, respectively. We write

$$(D_\infty)_{\mathrm{II}} = 2\lambda_\infty(D_\infty)_{\mathrm{I}} = 12\lambda_\infty\pi\mu cU,$$

where λ_∞ is the Stokes law correction to the drag on one of the spheres due to the presence of the other, in an infinite medium. Numerical values of λ_∞ as a function of h/c are provided by Stimson & Jeffery (1926) and, for the special case where the spheres touch, by Faxén (Oseen 1927, p. 162). Stimson & Jeffery's results have been accurately recalculated by the author to facilitate interpolation. They are given below in Table 1.

h/c	1·0	1·1276	1·5430	2·3525	3·7621	6·1322	10·0676
λ_∞	0·645	0·6596	0·7024	0·7677	0·8361	0·8915	0·9307

TABLE 1. Stokes law correction factor for two spheres in an infinite medium.

If U_{II} and $(U_\infty)_{\mathrm{II}}$ denote the terminal settling velocities of the two spheres in the bounded and unbounded fluid, respectively, we have that

$$\frac{(U_\infty)_{\mathrm{II}}}{U_{\mathrm{II}}} = \frac{1}{1-4{\cdot}209\lambda_\infty(c/l)}. \tag{4.1}$$

This relation cannot lead to correct results as $h/l \to \infty$, since the particles must then behave independently of one another. As c/l is already assumed to be small, this condition is equivalent to $h/c \to \infty$. Thus, when the particles behave independently, we have, from equation (3.2),

$$\lim_{h/c\to\infty} \frac{(U_\infty)_{\mathrm{II}}}{U_{\mathrm{II}}} = \frac{1}{1-2{\cdot}104(c/l)}. \tag{4.2}$$

It appears then, for a fixed value of c/l, starting from the point at which the spheres touch, that the wall effect should at first increase in magnitude with h/c, pass through a maximum and asymptotically approach equation (4.2) as a limit. As will appear shortly, this effect is observed experimentally (figure 2). This unusual behaviour may explain the 'kinks' observed by Kynch (1959) in his attempt to analyse the two-sphere settling data of Hall (1956), without reference to wall effects.

Pfeffer's (1958) experiments were conducted with pairs of identical $\frac{1}{4}$ and $\frac{1}{2}$ in. diameter spheres (nominal sizes) of materials of various density settling at the centre of a cylindrical column of $5\frac{11}{32}$ in. inside diameter. The experimental technique and results are reported by Happel & Pfeffer (1960), although only for the $\frac{1}{4}$ in. spheres. Only the data for the $\frac{1}{2}$ in. spheres will be examined here, as it is difficult with the smaller sizes to separate accurately the wall effect from the experimental inaccuracies.

The spheres used by Pfeffer were of Nylon, Lucite and Marbelette, their respective diameters being 0·499, 0·490 and 0·488 in. Only the data for the latter two sizes, which we characterize by the single diameter of 0·489 in., will be examined in detail. This makes $c/l = 0{\cdot}0915$. Experiments were carried out in the range from $h/c = 1{\cdot}0$ to about 6. Single sphere Reynolds numbers varied from 0·06 to 0·19. The settling velocities, U_{II}, were measured directly. Values of $(U_\infty)_{\mathrm{II}}$, required for testing the data, were obtained from the relation $(U_\infty)_{\mathrm{II}} = (U_\infty)_{\mathrm{I}}/\lambda_\infty$ by calculating the terminal settling velocity $(U_\infty)_{\mathrm{I}}$ of a single sphere in the unbounded fluid from the known properties of the spheres and liquid.

A direct comparison of the experimental and theoretical wall-effect ratios, $(U_\infty)_{\mathrm{II}}/U_{\mathrm{II}}$, as functions of h/c, is furnished in figure 2. The theoretical values, derived from equation (4.1) with $c/l = 0{\cdot}0915$, are shown by the solid curve. The dashed curve represents a smooth curve drawn through the experimental-data points. Also shown on the sketch is the theoretical asymptote, $(U_\infty)_{\mathrm{II}}/U_{\mathrm{II}} = 1{\cdot}239$ for the situation where $h/c \to \infty$. Theoretical and experimental values are concordant up to about $h/c = 2{\cdot}0$, beyond which the divergence between them appears significant. Particularly impressive is the agreement of the three experimental values of 1·318, 1·320 and 1·321 with the theoretical value of 1·330 for the case where the spheres touch.

5. Rotating particle in a bounded fluid

Equation (1.1) has an analogue for the couple on a particle rotating near a boundary. The couple $\mathbf{L}$ which must be applied to the particle to maintain it in uniform rotation with angular velocity $\mathbf{\Omega}$ is different from the corresponding couple, $\mathbf{L}_\infty$, in the absence of boundaries. The calculation of L/L_∞ can be made along lines similar to those laid out in §2.

In place of equation (2.3) and (2.8) it is now required that

$$\mathbf{v} = \mathbf{v}^{(1)} = \mathbf{\Omega} \times \mathbf{r} \quad \text{on } P. \tag{5.1}$$

The couple $\mathbf{L}$ is given by a sum of terms similar to equation (2.13), in which $\mathbf{L}^{(n)}$ replaces $\mathbf{D}^{(n)}$ (see Appendix), and in which $\mathbf{L}^{(1)} = \mathbf{L}_\infty$. The initial field, corresponding to the motion induced by a point couple situated at the centre of the particle, is (Love 1927, p. 187)

$$\mathbf{v}^{(1)} = -\mathbf{L}_\infty \times \mathbf{r}/8\pi\mu r^3, \quad p^{(1)} = 0. \tag{5.2}$$

When an arbitrary particle rotates in an unbounded fluid, it can be shown that there is a linear relation of the form

$$\mathbf{L}_\infty = -\mathbf{\Psi}.\mathbf{\Omega}, \tag{5.3}$$

where $\mathbf{\Psi}$ is a symmetric tensor. Every arbitrary particle therefore possesses a set of three mutually perpendicular principal axes of rotation.

As in §2, we next focus our attention on the calculation of $L^{(3)}$. In particular, we seek the generalization of a second law due to Faxén (Oseen 1927, p. 113) which, for a spherical particle, enables one to compute the couple acting upon it when suspended in an arbitrary field of flow. Towards this end, we again fall back on the reciprocal theorem, equation (2.17), which remains valid for the applications at hand. Now, on the surface of the particle, $\mathbf{v}^{(1)}$ is given by equation (5.1). The right-side of equation (2.17) is therefore equal to $\mathbf{\Omega}.\mathbf{L}^{(3)}$, where

$$\mathbf{L}^{(3)} = \iint_P \mathbf{r}\times(\mathbf{\Pi}^{(3)}.d\mathbf{S}).$$

To evaluate the left-hand integral of equation (2.17) we note, from equation (2.10), that $\mathbf{v}^{(3)} = -\mathbf{v}^{(2)}$ on P. As the field $\mathbf{v}^{(2)}$ is well behaved everywhere, we expand it in a Taylor series about the centre of the particle and retain only the leading terms, whence

$$\mathbf{v}^{(2)} = \mathbf{v}_0^{(2)} + \boldsymbol{\omega}_0^{(2)}\times\mathbf{r} + \mathbf{\Delta}_0^{(2)}.\mathbf{r}, \tag{5.4}$$

where $\boldsymbol{\omega}_0^{(2)} = \frac{1}{2}(\nabla\times\mathbf{v}^{(2)})_0$ and $\mathbf{\Delta}_0^{(2)} = \frac{1}{2}(\nabla\mathbf{v}^{(2)}+\mathbf{v}^{(2)}\nabla)_0$. This amounts to an expansion of the field into translational, rotational and dilatational contributions, respectively.

The left-side of equation (2.17) therefore becomes $-\mathbf{v}_0^{(2)}.\mathbf{D}^{(1)} - \boldsymbol{\omega}_0^{(2)}.\mathbf{L}_\infty - J$, where

$$\mathbf{D}^{(1)} = \iint_P d\mathbf{S}.\mathbf{\Pi}^{(1)}, \quad \mathbf{L}_\infty = \iint_P \mathbf{r}\times(\mathbf{\Pi}^{(1)}.d\mathbf{S}) \quad \text{and} \quad J = \iint_P d\mathbf{S}.\mathbf{\Pi}^{(1)}.\mathbf{\Delta}_0^{(2)}.\mathbf{r}. \tag{5.5}$$

It is obvious that the purely rotational field, $\mathbf{v}^{(1)}$, cannot produce any resultant force on the particle and, hence, $\mathbf{D}^{(1)} = \mathbf{0}$. It remains only to evaluate the surface integral J. At the risk of some loss in generality, we confine ourselves to the case in which $\mathbf{\Delta}_0^{(2)} = \mathbf{0}$, which makes $J = 0$. By considering the oddness and evenness of the components of $\mathbf{v}^{(2)}$, it can be shown that sufficient conditions for the vanishing of $\mathbf{\Delta}_0^{(2)}$ are: (i) the particle revolves about a principal axis of rotation, and (ii) the boundary, S, possesses three mutually perpendicular symmetry planes which are orthogonal to the principal axes of the particle. When these conditions are met, the directions of $\mathbf{L}$ and $\mathbf{L}_\infty$ are parallel to $\mathbf{\Omega}$.

Upon collecting results one obtains

$$\mathbf{\Omega}.\mathbf{L}^{(3)} = -\boldsymbol{\omega}_0^{(2)}.\mathbf{L}_\infty, \tag{5.6}$$

or, employing equation (5.3),

$$\mathbf{L}^{(3)} = \mathbf{\Psi}.\boldsymbol{\omega}_0^{(2)}. \tag{5.7}$$

This relation may be regarded as a generalization of Faxén's second law, to which it reduces for a spherical particle. In contrast to equation (2.19), when applied to an arbitrary particle, equation (5.7) is *not* generally correct unless the stipulated symmetry conditions are met.

Since $L^{(3)} = (L_\infty/\Omega)\,\omega_0^{(2)}$, we eventually obtain, as before,

$$\frac{L}{L_\infty} = \frac{1}{1-(\omega_0^{(2)}/\Omega)}. \tag{5.8}$$

Thus, by arguments similar to those of § 2,†

$$\frac{L}{L_\infty} = \frac{1}{1-K(L_\infty/8\pi\mu\Omega l^3)+O(c/l)^5}, \tag{5.9}$$

where K is a dimensionless constant of $O(1)$ which is independent of the shape of the particle.

It can be seen from this relation that the magnitude of the wall effect for a rotating particle depends on terms of $O(c/l)^3$, in contrast to the wall effect for a translating particle, which depends on terms of $O(c/l)$. Thus, the wall effect is very much smaller for rotation than for translation. Moreover, the rotational result applies to very much larger values of c/l than does the corresponding translational result.

In the case of a spherical particle, $L_\infty = 8\pi\mu c^3\Omega$, and equation (5.9) becomes

$$\frac{L}{L_\infty} = \frac{1}{1-K(c/l)^3+\dots}.$$

The K value for any particular surface S can be obtained by comparing the above to the known solution for a spherical particle. Only a very few solutions of the equations of motion are known for spheres rotating near boundaries. These are considered below.

Case I. Particle rotating at the centre of a hollow sphere; l = radius of sphere (Jeffery 1915):

$$K = 1. \tag{5.10}$$

Case II. Particle rotating about an axis which is perpendicular to a single, infinite plane surface; l = distance from centre of particle to plane (Jeffery 1915):

(*a*) 'solid' plane, $K = \frac{1}{8}$; (5.11)

(*b*) 'free' surface,‡ $K = -\frac{1}{8}$. (5.12)

Case III. Particle rotating about the longitudinal axis of a circular cylinder; l = radius of cylinder:

$$K = 0{\cdot}7968. \tag{5.13}$$

This result has not been given before.

Appendix

We prove here that a velocity field which is free from singularities in the interior of the volume, Q, occupied by a particle can produce neither a resultant force nor a couple on the particle. The force on the particle is

$$\mathbf{D} = \iint_P d\mathbf{S}\,.\,\mathbf{\Pi},$$

† The error estimate, $O(c/l)^5$, applies only to non-spherical particles. Spherical particles lead to much smaller errors.

‡ This is obtained from Jeffery's solution for the rotation, about their line-of-centres, of two equal spheres, external to each other in an infinite fluid, when they rotate at the same angular velocity in the same direction. The plane of symmetry midway between them is then a free surface.

which, since the field is regular in Q, can be converted by Gauss's divergence theorem into the volume integral

$$\mathbf{D} = -\iiint_Q \nabla . \mathbf{\Pi} \, dQ.$$

This, in turn, vanishes identically since $\nabla . \mathbf{\Pi} = \mathbf{0}$ in creeping flow.

The analogous proof for the couple on the particle is

$$\mathbf{L} = \iint_P \mathbf{r} \times (\mathbf{\Pi} . d\mathbf{S}) = -\iint_P d\mathbf{S} . (\mathbf{\Pi} \times \mathbf{r})$$
$$= \iiint_Q \nabla . (\mathbf{\Pi} \times \mathbf{r}) \, dQ = \iiint_Q (\nabla . \mathbf{\Pi}) \times \mathbf{r} \, dQ = \mathbf{0}. \qquad \text{Q.E.D.}$$

REFERENCES

Birkhoff, G. 1950 *Hydrodynamics*, p. 38. Princeton University Press.

Brenner, H. & Happel, J. 1958 *J. Fluid Mech.* **4**, 195.

Chang, I. D. 1961 *ZAMP*, **12**, 6.

Cunningham, E. 1910 *Proc. Roy. Soc.* A, **83**, 357.

Famularo, J. 1960 D.Eng. Sci. Thesis, New York University (unpublished).

Haberman, W. L. & Sayre, R. M. 1958 Motion of rigid and fluid spheres in stationary and moving liquids inside cylindrical tubes. *David Taylor Model Basin Report* no. 1143.

Hall, E. W. 1956 Ph.D. Thesis, Birmingham.

Happel, J. & Pfeffer, R. 1960 *J. Amer. Inst. Chem. Engrs*, **6**, 129.

Heiss, J. F. & Coull, J. 1952 *Chem. Engng Progr.* **48**, 133.

Jeffery, G. B. 1915 *Proc. Lond. Math. Soc.* (2), **14**, 327.

Kynch, G. J. 1959 *J. Fluid Mech.* **5**, 193.

Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics*, p. 67. Reading, Mass.: Addison Wesley Publ. Co.

Love, A. E. H. 1927 *A Treatise on the Mathematical Theory of Elasticity*, 4th ed. Cambridge University Press.

Oseen, C. W. 1927 *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Leipzig: Akademische Verlagsgesellschaft.

Péres, J. 1929 *C.R. Acad Sci., Paris*, **188**, 310.

Pettyjohn, E. S. & Christiansen, E. B. 1948 *Chem. Engng Progr.* **44**, 157.

Pfeffer, R. 1958 *The motion of two spheres following each other in a viscous fluid.* M.Ch.E. Thesis, New York University.

Squires, L. & Squires, W. 1937 *Trans. Amer. Inst. Chem. Engrs*, **33**, 1.

Stimson, M. & Jeffery, G. B. 1926 *Proc. Roy. Soc.* A, **111**, 110.

Villat, H. 1943 *Leçons sur les Fluides Visqueux*, p. 120. Paris: Gauthier-Villars.

Wakiya, S. 1957 *J. Phys. Soc. Japan*, **12**, 1130 (with corrigenda, *Ibid.* p. 1318).

Wakiya, S. 1959 Effect of a submerged object on a slow viscous flow-spheroid at an arbitrary angle of attack. *Research Report* no. 8, *College of Engineering, Niigata Univ.* (in Japanese).